Wigner's D-matrix elements for $SU(3)$ - A Generating Function Approach

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(January 1996)

Abstract

A generating function for the Wigner's D-matrix elements of $SU(3)$ is derived. From this an explicit expression for the individual matrix elements is obtained in a closed form.

PACS number(s):

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I. INTRODUCTION

The Wigner's D-matrix elements of $SU(3)$ have very important applications in nuclear physics, particle physics, $SU(3)$ lattice gauge theories, matrix models, finite temparature field theory calculations involving $SU(3)$ and other areas of physics. Starting with Murnaghan [2], who parametrized the defining matrices of $U(n)$ and $O(n)$, many authors [4–6] have obtained expressions for the Wigner's D-matrix elements of $SU(3)$ using various methods. It is the purpose of this paper to evaluate these matrix elements for $SU(3)$ using the calculus we [1] have set up to deal with computations involving the group $SU(3)$. The distinct advantage of this calculus and the novelty of our present method is that it allows one to write a generating function for these matrix elements from which one can extract the individual matrix elements by using the auxiliary inner product of the calculus.

The plan of the paper is as follows. We begin, in section 2, by reviewing the main ingredients of our calculus for $SU(3)$ which are relevant to our present discussion and then, in section 3, give a derivation of the generating function for the matrix elements. In section 4 we show how to extract the individual matrix elements and obtain a polynomial expression for the matrix elements in any irreducible representation in terms of the matrix elements of the defining representation of $SU(3)$ in any parametrization. Section 5 is devoted to a discussion of our results. A few examples are included in the appendix for illustrating the method.

II. OVERVIEW OF OUR PREVIOUS RESULTS

In this section we briefly review the results that we need on the group $SU(3)$. Some of these results were obtained by us in a previous paper [1].

$SU(3)$ is the group of 3×3 unitary unimodular matrices A with complex coefficients. It is a group of 8 real parameters. The matrix elements satisfy the following conditions

$$A = (a_{ij}),$$

$$A^\dagger A = I, \qquad AA^\dagger = I\,, \text{ where } I \text{ is the identity matrix and },$$
$$\det(A) = 1\,. \tag{1}$$

A. Parametrization

One well known parametrization of $SU(3)$ is due to Murnaghan [2], see also [3–5,8]. In this we write a typical element of $SU(3)$ as :

$$D(\delta_1, \delta_2, \phi_3) U_{23}(\phi_2, \sigma_3) U_{12}(\theta_1, \sigma_2) U_{13}(\phi_1, \sigma_1)\,, \tag{2}$$

with the condition $\phi_3 = -(\delta_1 + \delta_2)$. Here D is a diagonal matrix whose elements are $\exp(i\delta_1)$, $\exp(i\delta_2)$,$\exp(i\phi_3)$ and $U_{pq}(\phi,\sigma)$ is a 3×3 unitary unimodular matrix which for instance in the case $p = 1$, $q = 2$ has the form

$$\begin{pmatrix} \cos\phi & -\sin\phi\exp(-i\sigma) & 0 \\ \sin\phi\exp(i\sigma) & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

The 3 parameters ϕ_1, ϕ_2, ϕ_3 are longitudinal angles whose range is $-\pi \le \phi_i \le \pi$, and the remaining 6 parametrs are latitude angles whose range is $\frac{1}{2}\pi \le \sigma_i \le \frac{1}{2}\pi$.

Now the trnasformations U_{23} and U_{13} can be changed into transformations of the type U_{12} whose matrix elements are known, by the following device

$$\begin{aligned} U_{13}(\phi_1, \sigma_1) &= (2,3) U_{12}(\phi_1, \sigma_1)(2,3)\,, \\ U_{23}(\phi_2, \sigma_3) &= (1,2)(2,3) U_{12}(\phi_2, \sigma_3)(2,3)(1,2)\,, \end{aligned} \tag{4}$$

where $(1,2)$ and $(2,3)$ are the transposition matrices

$$(1,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2,3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

In this way the expression for an element of the $SU(3)$ group becomes

$$D(\delta_1, \delta_2, \phi_3)(1,2)(2,3) U_{12}(\phi_2, \sigma_3)(2,3)(1,2) U_{12}(\theta_1, \sigma_2)(2,3) U_{12}(\phi_1, \sigma_1)(2,3)\,. \tag{6}$$

B. Irreducible Representations.

The above parametrization provides us with a defining irreducible representation $\underline{3}$ of $SU(3)$ acting on a 3 dimensional complex vector space spanned by the triplet z_1, z_2, z_3 of complex variables. The hermitian adjoint of the above matrix gives us another defining but inequivalent irreducible representaion $\underline{3^*}$ of $SU(3)$ acting on the triplet w_1, w_2, w_3 of complex variables spanning another 3 dimensional complex vector space. Tensors constructed out of these two 3 dimensional representations span an infinite dimensional complex vector space.

C. The Constraint

If we impose the constraint

$$z_1 w_1 + z_2 w_2 + z_3 w_3 = 0\,, \tag{7}$$

on this space we obtain an infinite dimensional complex vector space in which each irreducible representation of $SU(3)$ occurs once and only once. Such a space is called a model space for $SU(3)$. Further if we solve the constraint $z_1 w_1 + z_2 w_2 + z_3 w_3 = 0$ and eliminate one of the variables, say w_3, in terms of the other five variables z_1, z_2, z_3, w_1, w_2 we can write a genarating function to generate all the basis states of all the IRs of $SU(3)$. This generating function is computationally a very convenient realization of the basis of the model space of $SU(3)$. Moreover we can define a scalar product on this space by choosing one of the variables, say z_3, to be a planar rotor $\exp(i\theta)$. Thus the model space for $SU(3)$ is now a Hilbert space with this('auxiliary') scalar product between the basis states. The above construction was carried out in detail in a previous paper by us [1]. For easy accessability we give a self-contained summary of those results here.

D. Labels for the basis states.

(i). Gelfand-Zetlein labels

Normalized basis vectors are denoted by, $|M, N; P, Q, R, S, U, V>$. All labels are non-negative integers. All Irreducible Represenatations(IRs) are uniquely labeled by (M, N). For a given IR (M, N), labels (P, Q, R, S, U, V) take all non-negative integral values subject to the constraints:

$$R + U = M \quad , \quad S + V = N \quad , \quad P + Q = R + S. \tag{8}$$

The allowed values can be presribed easily: R takes all values from 0 to M, and S from 0 to N. For a given R and S, Q takes all values from 0 to $R + S$.

(ii). Quark model labels.

The relation between the above Gelfand-Zetlein labels and the Quark Model labels is as given below.

$$2I = P + Q = R + S,\ 2I_3 = P - Q,\ Y = \frac{1}{3}(M - N) + V - U = \frac{2}{3}(N - M) - (S - R)\,. \tag{9}$$

where R takes all values from 0 to M. S takes all values from 0 to N. For a given R and S, Q takes all values from 0 to $R + S$.

E. Explicit realization of the basis states

(i). Generating function for the basis states of $SU(3)$

The generating function for the basis states of the IR's of $SU(3)$ can be written as

$$g(p, q, r, s, u, v) = \exp(r(pz_1 + qz_2) + s(pw_2 - qw_1) + uz_3 + vw_3)\,. \tag{10}$$

The coefficient of the monomial $p^P q^Q r^R s^S u^U v^V$ in the Taylor expansion of Eq.(10), after eliminating w_3 using Eq.(7), in terms of these monomials gives the basis state of $SU(3)$ labelled by the quantum numbers P, Q, R, S, U, V.

(ii). Formal generating function for the basis states of $SU(3)$

The generating function Eq.(10) can be written formally as

$$g = \sum_{P,Q,R,S,U,V} p^P q^Q r^R s^S u^U v^V |PQRSUV) \,. \tag{11}$$

where $|PQRSTUV)$ is an unnormalized basis state of $SU(3)$ labelled by the quantum numbers P, Q, R, S, U, V.

Note that the constraint $P + Q = R + S$ is automatically satisfied in the formal as well as explicit Taylor expansion of the generating function.

(iii). Generalized generating function for the basis states of $SU(3)$

It is useful, while computing the normalizations(see below) of the basis states, to write the above generating function in the following form

$$\mathcal{G}(p, q, r, s, u, v) = \exp(r_p z_1 + r_q z_2 + s_p w_2 + s_q w_1 + u z_3 + v w_3) \,. \tag{12}$$

In the above generalized generating function (12) the following notation holds.

$$r_p = rp, \qquad r_q = rq, \qquad s_p = sp, \qquad s_q = -sq \,. \tag{13}$$

F. Notation

Hereafter, for simplicity in notation we assume all variables other than the z^i_j and w^i_j where $i, j = 1, , 2, 3$ are real eventhough we have treated them as comlex variables at some places. Our results are valid even without this restriction as we are interested only in the coefficients of the monomials in these real variables rather than in the monomials themselves.

G. 'Auxiliary' scalar product for the basis states.

The scalar product to be defined in this section is 'auxiliary' in the sense that it does not give us the 'true' normalizations of the basis astes of $SU(3)$. However it is compuataionally

very convenient for us as all compuattaions with this scalar product get reduced to simple Gaussian integrations and the 'true' normalizations themselves can then be got quite easily.

(i). Scalar product between generating functions of basis states of $SU(3)$

We define the scalar product between any two basis states in terms of the scalar product between the corresponding generating functions as follows :

$$(g', g) = \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} \int \frac{d^2 z_1}{\pi^2} \frac{d^2 z_2}{\pi^2} \frac{d^2 w_1}{\pi^2} \frac{d^2 w_2}{\pi^2} \exp(-\bar{z}_1 z_1 - \bar{z}_2 z_2 - \bar{w}_1 w_1 - \bar{w}_2 w_2)$$

$$\times \exp((r'(p' z_1 + q' z_2) + s'(p' w_2 - q' w_1) - \frac{-v'}{z_3}(z_1 w_1 + z_2 w_2) + u' \bar{z}_3)$$

$$\times \exp((r(p z_1 + q z_2) + s(p w_2 - q w_1) - \frac{-v}{z_3}(z_1 w_1 + z_2 w_2) + u z_3)\,,$$

$$= (1 - v'v)^{-2} \left(\sum_{n=0}^{\infty} \frac{(u'u)^n}{(n!)^2} \right) \exp\left[(1 - v'v)^{-1} (p'p + q'q)(r'r + s's) \right]. \tag{14}$$

(ii). Choice of the variable z_3

To obtain the Eq.(14 we have made the choice

$$z_3 = \exp(i\theta)\,. \tag{15}$$

The choice, Eq.(15), makes our basis states for $SU(3)$ depened on the variables z_1, z_2, w_1, w_2 and θ.

(iii). Scalar product between the gneralized generating functions of the basis states of $SU(3)$

For the generalized generating function the scalar product becomes

$$(\mathcal{G}', \mathcal{G}) = (1 - v'v)^{-2} \exp\left[(1 - v'v)^{-1} ({r_p}' r_p + {r_q}' r_q + {s_p}' s_p + {s_q}' s_q) \right]$$

$$\times \left[\sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(u' - v \frac{({r_p}' {s_q}' + {r_q}' {s_p}')}{(1 - v'v)} \right)^n \cdot \left(u - v' \frac{(r_p s_q + r_q s_p)}{(1 - v'v)} \right)^n \right], \tag{16}$$

and as in Eq.(13)

$$r_p = rp, \qquad r_q = rq, \qquad s_p = sp, \qquad s_q = -sq\,,$$
$$r'_p = r'p', \qquad r'_q = r'q', \qquad s'_p = s'p', \qquad s'_q = -s'q'\,. \tag{17}$$

H. Normalizations

(i). 'Auxiliary' normalizations of unnormalized basis states

The scalar product between two unnormalized basis states, computed using our 'auxiliary scalar product, is given by,

$$M(PQRSUV) \equiv (PQRSUV|PQRSUV)\,,$$
$$= \frac{(V+P+Q+1)!}{P!Q!R!S!U!V!(P+Q+1)}\,. \tag{18}$$

(ii). Scalar product between the unnormalized and normalized basis states

The scalar product, computed using our 'auxiliary' scalar product, between an unnormalized basis state and a normalized one is given by the next equation where it is denoted by $(PQRSUV||PQRSUV>$.

$$(PQRSUV||PQRSUV >= N^{-1/2}(PQRSUV) \times M(PQRSUV) \tag{19}$$

(iii). 'True' normalizations of the basis states

We call the ratio of the 'auxiliary' norm of the unnormalized basis sate represented by $|PQRSUV)$ and the scalar product of the unnormalized basis state with a normalized Gelfand-Zeitlin state, represented by $|PQRSUV>$, as 'true' normalization. It is given by

$$N^{1/2}(PQRSUV) \equiv \frac{(PQRSUV|PQRSUV)}{< PQRSUV|PQRSUV >}$$
$$= \left(\frac{(U+P+Q+1)!(V+P+Q+1)!}{P!Q!R!S!U!V!(P+Q+1)}\right)^{1/2}\,. \tag{20}$$

III. GENERATING FUNCTION FOR THE WIGNER'S D-MATRIX ELEMENTS OF $SU(3)$.

$$g(p,q,r,s,u,v,z_1,z_2,w_1,w_2) = \sum_{P,Q,R,S,U,V} p^P q^Q r^R s^S u^U v^V |PQRSUV) \,, \qquad (21)$$

where $|PQRSUV)$ is an unnormalized basis state in the IR labeled by the two positive integers $(M = R+U, N = S+V)$.

We know from Eq.(20),

$$|PQRSUV) = N^{(1/2)}(PQRSUV)|PQRSUV> \,, \qquad (22)$$

where $2I = P+Q$ and $|PQRSUV)$ is a normalized basis state.

Therefore

$$g = \sum_{PQRSUV} \left(\frac{(U+2I+1)!(V+2I+1)!}{P!Q!R!S!U!V!(2I+1)} \right)^{(1/2)} p^P q^Q r^R s^S u^U v^V |PQRSUV> \ . \qquad (23)$$

Now,

$$Ag(p,q,...) = \sum_{PQRSUV} \sum_{P'Q'R'S'U'V'} \left(\frac{(U+2I+1)!(V+2I+1)!}{P!Q!R!S!U!V!(2I+1)} \right)^{(1/2)}$$

$$\times D^{(M=R+U,\ N=S+V)}_{PQRSUV,\ P'Q'R'S'U'V'} \times \ p^P q^Q r^R s^S u^U v^V \times \ |PQRSUV> \ . \qquad (24)$$

To get a generating function for the matrix elements alone we have to take the inner product of this transformed generating function with the generating function for the basis states. Thruoghout the following we take the variables p, q, r, s, u, v together with their primed and unprimed variants to be real since we are interested only in the coefficients of monomials in these different sets of varibles in different expansions and are not interested in these variables or their functions as such.

Thus,

$$(g(p", q", r", s", u", v";\ z_1, z_2, z_3, w_1, w_2),\ Ag(p,q,r,s,u,v;\ z_1, z_2, z_3, w_1, w_2))$$

$$= \sum_{PQRSUV} \sum_{P'Q'R'S'U'V'} \sum_{P"Q"R"S"U"V"} \left(\frac{(U+2I+1)!(V+2I+1)!}{P!Q!R!S!U!V!(2I+1)}\right)^{(1/2)}$$

$$\times (P"Q"R"S"U"V" \| P'Q'R'S'U'V' > \times D^{(M=R+U,\ N=S+V)}_{PQRSUV,\ P'Q'R'S'U'V'}(A)$$

$$\times p^P q^Q r^R s^S u^U v^V p"^{P"} q"^{Q"} r"^{R"} s"^{S"} u"^{U"} v"^{V"} \ . \tag{25}$$

But we know from Eq.(19),

$$(\ P"Q"R"S"U"V" || P'Q'R'S'U'V' >$$

$$= \left(\frac{(U'+2I'+1)!(V'+2I'+1)!}{P'!Q'!R'!S'!U'!V'!(2I'+1)}\right)^{(-1/2)} \times \frac{(V'+P'+Q'+1)!}{P'!Q'!R'!S'!U'!V'!(P'+Q')}$$

$$\times \delta_{P"P'} \delta_{Q"Q'} \delta_{R"R'} \delta_{S"S'} \delta_{U"U'} \delta_{V"V'} \ . \tag{26}$$

Substituting this formula and changing the double primed variables to single primed ones, we get

$$(g(p', q', r,' s', u', v'; \ z_1, z_2, z_3, w_1, w_2), \quad Ag(p, q, r, s, u, v; \ z_1, z_2, z_3, w_1, w_2))$$

$$= \sum_{PQRSUV;\ P'Q'R'S'U'V'} \left(\frac{(U+2I+1)!(V+2I+1)!}{P!Q!R!S!U!V!(2I+1)}) \times (\frac{P'!Q'!R'!S'!U'!V'!(2I'+1)}{(U'+2I'+1)!(V'+2I'+1)!}\right)^{(1/2)}$$

$$\times (\frac{(V'+P'+Q'+1)!}{P'!Q'!R'!S'!U'!V'!(P'+Q'+1)}) \times D^{(M=R+U,\ N=S+V)}_{PQRSUV,\ P'Q'R'S'U'V'}(A)$$

$$\times p^P q^Q r^R s^S u^U v^V p'^{P'} q'^{Q'} r'^{R'} s'^{S'} u'^{U'} v'^{V'} \ . \tag{27}$$

We therefore conclude that the Wigner's D-matrix element,

$$D^{(M=R+U,\ N=S+V)}_{PQRSUV,\ P'Q'R'S'U'V'}$$

for $SU(3)$ can be obtained as the coefficient of the monomial,

$$p^P q^Q r^R s^S u^U v^V \times p'^{P'} q'^{Q'} r'^{R'} s'^{S'} u'^{U'} v'^{V'} \ ,$$

multiplied by,

$$\left(\frac{P!Q!R!S!U!V!(2I+1)}{(U+2I+1)!(V+2I+1)!} \times \frac{(U'+2I'+1)!(V'+2I'+1)!}{P'!Q'!R'!S'!U'!V'!(2I'+1)} \right)^{(1/2)}$$

$$\times (\frac{P'!Q'!R'!S'!U'!V'!(P'+Q'+1)}{(V'+P'+Q'+1)!}) \ , \tag{28}$$

in the inner product $(g', \ Ag)$ between the untransformed and transformed generating functions for the basis states.

Next we calculate this inner product using the explicit realization for the generating function. For this purpose it is advantageous, as will be seen in a minute, to use the generalized generating function for the basis states

$$\mathcal{G} = \exp(r_p z_1 + r_q z_2 + s_p w_2 + s_q w_1 + u z_3 + v w_3)$$

$$= \exp\left(\begin{pmatrix} r_p & r_q & u \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} s_q \\ s_p \\ v \end{pmatrix} \right) \ . \tag{29}$$

When any element $A \in SU(3)$ acts on this generating function it undergoes the following transformation

$$A\mathcal{G} = \exp\left(\begin{pmatrix} r_p & r_q & u \end{pmatrix} A \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} A^\dagger \begin{pmatrix} s_q \\ s_p \\ v \end{pmatrix} \right) \ . \tag{30}$$

As is clear from the above equation we can let the triplets r_p, r_q, u and s_q, s_p, v undergo the transformation instead of the triplets z_1, z_2, z_3 and w_1, w_2, w_3. Therfore we can write the transformed generating function as

$$A\mathcal{G} = \mathcal{G}(r_p", r_q", u"; s_q", s_p", v") \ , \tag{31}$$

where

$$r_p" = a_{11}r_p + a_{21}r_q + a_{31}u$$

$$r_q" = a_{12}r_p + a_{22}r_q + a_{32}u$$

$$u" = a_{13}r_p + a_{23}r_q + a_{33}u\,,$$

$$s_q" = a^*_{11}s_q + a^*_{21}s_p + a^*_{31}v$$

$$s_p" = a^*_{12}s_q + a^*_{22}s_p + a^*_{32}v$$

$$v" = a^*_{13}s_q + a^*_{23}s_p + a^*_{33}v\,. \tag{32}$$

To continue with our computation we have to take the inner product of this transformed generating function with the (untransformed) generating function of the basis states.

This is known to us from Eq.(16) as

$$(\mathcal{G}', \mathcal{G}") = (1 - v'v")^{-2}\text{exp}\left[(1 - v'v")^{-1}(r_p'r_p" + r_q'r_q" + s_p's_p" + s_q's_q")\right]$$

$$\times \left[\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(u' - v"\frac{(r_p's_q' + r_q's_p')}{(1 - v'v")}\right)^n\left(u" - v'\frac{(r_p"s_q" + r_q"s_p")}{(1 - v'v")}\right)^n\right]\,. \tag{33}$$

This expression gets further simplified if we substitute from Eq.(13)

$$r_p' = r'p', \quad r_q' = r'q', \quad s_q' = -s'q', \quad s_p' = s'p\,.$$

We, therefore, get

$$(\mathcal{G}', \mathcal{G}") = (1 - v'v")^{-2}\text{exp}\left[(1 - v'v")^{-1}(r_p'r_p" + r_q'r_q" + s_p's_p" + s_q's_q")\right]$$

$$\times \left[\sum_{n=0}^{\infty}\frac{1}{(n!)^2}(u')^n(u" - v'\frac{(r_p"s_q" + r_q"s_p")}{(1 - v'v")})^n\right]\,. \tag{34}$$

One last simplification can be brought about in the above expression when we recognize that

$$\begin{aligned} r_p"s_q" + r_q"s_p" + u"v" &= r_ps_q + r_qs_p + vu\,, \\ &= \qquad vu\,. \end{aligned} \tag{35}$$

This tells us that

$$r_p"s_q" + r_q"s_p" = uv - u"v" \ . \tag{36}$$

Substituting this in our expression Eq.(34) for the inner product we get,

$$(\mathcal{G}', \mathcal{G}") = (1 - v'v")^{-2}\text{exp}\left[(1 - v'v")^{-1}(r_p'r_p" + r_q'r_q" + s_p's_p" + s_q's_q")\right]$$

$$\times \left[\sum_{n=0}^{\infty}\frac{1}{(n!)^2}(u')^n(u" - v'\frac{(uv - u"v")}{(1 - v'v")})^n\right]$$

$$= (1 - v'v")^{-2}\text{exp}\left[(1 - v'v")^{-1}(r_p'r_p" + r_q'r_q" + s_p's_p" + s_q's_q")\right]$$

$$\times \left[\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(u'\frac{(u" - uvv')}{(1 - v'v")}\right)^n\right] \ . \tag{37}$$

On the other hand if we use our present slightly modified scalar product then,

$$(\mathcal{G}', \mathcal{G}") = (1 - v'v")^{-2}\text{exp}\left[\frac{(r_p'r_p" + r_q'r_q" + s_p's_p" + s_q's_q")}{(1 - v'v")}\right.$$

$$\left. + \left(u' - v"\frac{(r_p's_q' + r_q's_p')}{(1 - v'v")}\right)\left(u" - v'\frac{(r_p"s_q" + r_q"s_p")}{(1 - v'v")}\right)\right] ,$$

$$= (1 - v'v")^{-2}\text{exp}\left[\frac{(r_p'r_p" + r_q'r_q" + s_p's_p" + s_q's_q") + u'(u" - uvv')}{(1 - v'v")}\right] \tag{38}$$

The expression on the right hand side of Eq.(37) or of Eq.(38) is our generating function for the Wigner's D-matrix elements of $SU(3)$.

IV. WIGNER'S D-MATRIX ELEMENTS OF $SU(3)$ IN ANY IRREDUCIBLE REPRESENTATION.

In this section our task is to extract the coefficient of the mononial

$$p^P q^Q r^R s^S u^U v^V \times p'^{P'} q'^{Q'} r'^{R'} s'^{S'} u'^{U'} v'^{V'} \ .$$

in the expansion of the generating function that we have obtained above, Eq.(37), for the Wigner's D-matrix elements of $SU(3)$. For this purpose we expand the right hand side of the above generating function and obtain

$$\sum_{m=0}^{\infty} \frac{(r_p{}'r_p" + r_q{}'r_q" + s_p{}'s_p" + s_q{}'s_q")}{m!(1-v'v")^m} \times \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(u' \frac{(u" - uvv')}{(1-v'v")^{(1+2/n)}} \right)^n ,$$

$$= \sum_{m=n=s=0}^{\infty} \sum_{t,m_1,m_2,m_3=0}^{n,m,m-m_1,m-m_1-m_2} \frac{(s+m+n+1)!}{m!n!(n-t)!t!(m+n+1)!s!(m-m_1-m_2-m_3)!m_3!}$$

$$\times (r_p")^{m_1}(r_q")^{m_2}(s_p")^{m_3}(s_q")^{m-\sum m_i}(p')^{m_1+m_3}(q')^{m-m_1-m_3}(r')^{m_1+m_2}(s')^{m-m_1-m_2}$$

$$\times (u')^n (v')^{n-t+s} u'^n v'^{n-t+s} u"^t v"^s (-uv)^{n-t} . \qquad (39)$$

Now let,

$$\begin{aligned} m_1 + m_2 &= P' , \\ m - m_1 - m_3 &= Q' , \\ m_1 + m_2 &= R' , \\ m - m_1 - m_2 &= S' , \\ n &= U' , \\ n - t + s &= V' . \end{aligned} \qquad (40)$$

The above assignments imply,

$$\begin{aligned} m &= P' + Q' \\ m_2 - m_3 &= R' - P' \\ m_2 &= R' - P' + m_3 , \\ \text{and,} \qquad s &= t + V' - U' . \end{aligned} \qquad (41)$$

This gives us

$$D^{(M=R+U,\ N=S+V)}_{PQRSUV,\ P'Q'R'S'U'V'}(A)$$

$$= \sum_{P'+Q'=0}^{\infty} \sum_{U'=0}^{\infty} \sum_{V'=U'}^{\infty} \sum_{m_1=0}^{P'+Q} \sum_{t=0}^{U'} \sum_{m_3=0}^{S'} \sum_{m_2=0}^{P'+Q'-m_1}$$

$$\times \frac{(t+V'-U')!(-uv)^{U'-t}}{(P'+Q')!U'!(U'-t)!t!(P'+Q'+U'+1)!(t+V'-U')!(R'-P'+m_3)!(S'-m_3)!m_3!}$$

$$\times (r_p")^{m_1}(r_q")^{m_2}(s_p")^{m_3}(s_q")^{S'-m_3}u"^{t}v"^{t+V'-U'} \times (p')^{P'}(q')^{Q'}(r')^{R'}(s')^{S'}(u')^{U'}(v')^{V'}\ . \tag{42}$$

In the above we substitute for the following from Eq.(32)

$$r_p",\quad r_q",\quad u",\quad s_q",\quad s_p",\quad v"\ ,$$

and get,

$$D^{(M=R+U,\ N=S+V)}_{PQRSUV,\ P'Q'R'S'U'V'}(A)$$

$$= \sum_{P'+Q'=0}^{\infty} \sum_{U'=0}^{\infty} \sum_{V'=U'}^{\infty} \sum_{m_1=0}^{P'+Q} \sum_{t=0}^{U'} \sum_{m_3=0}^{S'} \sum_{m_2=0}^{P'+Q'-m_1} \sum_{m_{11}+m_{12}+m_{13}=m_1} \sum_{m_{21}+m_{22}+m_{23}=m_2}$$

$$\sum_{m_{31}+m_{32}+m_{33}=m_3} \sum_{m_{41}+m_{42}+m_{43}=S'-m_3} \sum_{t_{11}+m_{12}+m_{13}=t} \sum_{t_{21}+t_{22}+t_{23}=t+V'-U'}$$

$$\frac{(-1)^{U'-t}(t+V'-U')!(-uv)^{U'-t}}{(P'+Q')!U'!(U'-t)!t!(P'+Q'+U'+1)!(t+V'-U')!(R'-P'+m_3)!(S'-m_3)!m_3!}$$

$$\times \frac{m_1!m_2!m_3!(S'-m_3)!t!(t+V'-U')!}{m_{11}!m_{12}!m_{13}!m_{21}!m_{22}!m_{23}!m_{31}!m_{32}!m_{33}!m_{41}!m_{42}!m_{43}!t_{11}!t_{12}!t_{13}!t_{21}!t_{22}!t_{23}!}$$

$$\times (a_{11})^{m_{11}}(a^*_{11})^{m_{11}}(a_{21})^{m_{12}}(a^*_{21})^{m_{42}}(a_{31})^{m_{13}}(a^*_{31})^{m_{43}}(a_{12})^{m_{21}}(a^*_{12})^{m_{31}}(a_{22})^{m_{22}}(a^*_{22})^{m_{32}}(a_{32})^{m_{23}}$$

$$\times (a^*_{32})^{m_{33}}(a_{13})^{t_{11}}(a^*_{13})^{t_{21}}(a_{23})^{t_{12}}(a^*_{23})^{t_{22}}(a_{33})^{t_{13}}(a^*_{33})^{t_{23}}$$

$$\times(p')^{P'}(q')^{Q'}(r')^{R'}(s')^{S'}(u')^{U'}(v')^{V'} \times (p)^{P}(q)^{Q}(r)^{R}(s)^{S}(u)^{U}(v)^{V} . \tag{43}$$

where we have made the identifications,

$$\begin{aligned}
m_{11}+m_{21}+t_{11}+m_{32}+m_{42}+t_{22} &= P\,,\\
m_{12}+m_{22}+t_{12}+m_{41}+m_{32}+t_{22} &= Q\,,\\
m_{11}+m_{21}+t_{11}+m_{12}+m_{22}+t_{22} &= R\,,\\
m_{41}+m_{31}+t_{21}+m_{42}+m_{32}+t_{22} &= S\,,\\
m_{13}+m_{23}+t_{13}+U'-t &= U\,,\\
m_{43}+m_{33}+t_{23}+U'-t &= V\,.
\end{aligned} \tag{44}$$

Finally, we get the desired object i.e., the Wigner's D-matrix or the finite transformation matrix of the group $SU(3)$ in any irreducible representation by multiplying the above matrix element by the factor in Eq.(28).

So finally,

$$D^{(M=R+U,\ N=S+V)}_{PQRSUV,\ P'Q'R'S'U'V'}(A)$$

$$= \left(\frac{P!Q!R!S!U!V!(2I+1)}{(U+2I+1)!(V+2I+1)!} \times \frac{(U'+2I'+1)!(V'+2I'+1)!}{P'!Q'!R'!S'!U'!V'!(2I'+1)}\right)^{(1/2)}$$

$$\times\left(\frac{P'!Q'!R'!S'!U'!V'!(P'+Q'+1)}{(V'+P'+Q'+1)!}\right)$$

$$\times \sum_{P'+Q'=0}^{\infty} \sum_{U'=0}^{\infty} \sum_{V'=U'}^{\infty} \sum_{m_1=0}^{P'+Q} \sum_{t=0}^{U'} \sum_{m_3=0}^{S'} \sum_{m_2=0}^{P'+Q'-m_1} \sum_{m_{11}+m_{12}+m_{13}=m_1} \sum_{m_{21}+m_{22}+m_{23}=m_2}$$

$$\sum_{m_{31}+m_{32}+m_{33}=m_3} \sum_{m_{41}+m_{42}+m_{43}=S'-m_3} \sum_{t_{11}+m_{12}+m_{13}=t} \sum_{t_{21}+t_{22}+t_{23}=t+V'-U'}$$

$$\times \frac{(-1)^{U'-t}(t+V'-U')!(-uv)^{U'-t}}{(P'+Q')!U'!(U'-t)!t!(P'+Q'+U'+1)!(t+V'-U')!(R'-P'+m_3)!(S'-m_3)!m_3!}$$

$$\times \frac{m_1!m_2!m_3!(S'-m_3)!t!(t+V'-U')!}{m_{11}!m_{12}!m_{13}!m_{21}!m_{22}!m_{23}!m_{31}!m_{32}!m_{33}!m_{41}!m_{42}!m_{43}!t_{11}!t_{12}!t_{13}!t_{21}!t_{22}!t_{23}!}$$

$$\times (a_{11})^{m_{11}}(a^*_{11})^{m_{11}}(a_{21})^{m_{12}}(a^*_{21})^{m_{42}}(a_{31})^{m_{13}}(a^*_{31})^{m_{43}}(a_{12})^{m_{21}}(a^*_{12})^{m_{31}}(a_{22})^{m_{22}}(a^*_{22})^{m_{32}}(a_{32})^{m_{23}}$$

$$\times (a^*_{32})^{m_{33}}(a_{13})^{t_{11}}(a^*_{13})^{t_{21}}(a_{23})^{t_{12}}(a^*_{23})^{t_{22}}\ (a_{33})^{t_{13}}(a^*_{33})^{t_{23}}\Big)\ . \tag{45}$$

The above equation, Eq.(45) for the Wigner's D-matrix element for $SU(3)$ is the analogue of Wigner's D-matrix element for $SU(2)$(see for example [9,10]).

V. DISCUSSION.

In this paper, making use of the tools of a calculus that we had set up previously to do computations on $SU(3)$, we have obtained (i) a generating function(Eq.(37),(38)) for the Wigner's D-matrix elements of $SU(3)$ and (ii) a closed form algebraic expression(Eq.(45)) for the individual Wigner's D-matrix elements of $SU(3)$ in any irreducible representation. To our knowledge this is the first time that a such generating function has been written for $SU(3)$. But this generating function gives us unitary matrix elements of $SU(3)$ only up to a multiplicative factor. The reason for this is that our auxiliary measure for the basis states is not a group invariant measure. This is clearly a drawback. However for computing objects such as the group characters this is no hurdle since the characters are invariant under basis transformtions.

We also note that our generating function is in fact a product of two factors one of which is an exponential function and the second is some power series. This seems to be a consequence of the particular choice of variables occuring in the construction of our basis functions which makes it possible for the θ variable part of any object of interest, such as for example the Clebsch-Gordan coefficients etc, to decouple from the part that dependeds on other variables.

Next, the expression for the individual D-matrix elements for $SU(3)$ has been obtained by many people previously also [4–6]. But one desirable feature about our expression is that it is quite compact and is independent of any particular parametrization used for describing the defining representation of $SU(3)$. Now coming back to the generating function for the matrix elements we recall, from our experience in computing the Clebsch-Gordan coefficients of $SU(3)$ previously and now the present computation of D-matrix elements that problems which are intractable by other methods may be, some times, easier to deal with using the generating function method. Therefore it is hoped that, now that a calculus and a generating function for Wigner's D-matrix elements are available, one might be able to employ this technique profitably to problems of interest in some areas of physics.

Appendix : Examples To compute the matrix elements of $SU(3)$, for lower dimensions, it is easier to work with the generating function for the matrix elements Eq.(37), (38)).

For the irreducible representation $\underline{3}$ the only terms of the generating function which are relavent are the ones linear in the primed and and doubly primed composite variables $r'_p, r''_p \cdots$. This gives us the following expansion for the generating function

$$r'_p r''_p + r'_q r''_q + s'_p s''_p + s'_q s''_q + u' u'' + v' v'' \,. \tag{A.1}$$

We now substitute for the doubly primed variables, in the above expression, from the Eqs.((32), (17)), and extract the coefficients of the various monomials $p^P q^Q r^R s^S u^U v^V$ for the values of the quantum numbers P, Q, R, S, U, V given in the table below for the IR $\underline{3}$. This gives us the $SU(3)$ representative matrix Eq(1).

$\underline{3}(M{=}1,\ N{=}0)$

.	P	Q	R	S	U	V	I	I_3	Y	$\|PQRSTUV)$	$N^{1/2}$
u	1	0	1	0	0	0	1/2	1/2	1/3	z_1	$\sqrt{2}$
d	0	1	1	0	0	0	1/2	$-1/2$	1/3	z_2	$\sqrt{2}$
s	0	0	0	0	1	0	0	0	-2/3	z_3	$\sqrt{2}$

$\underline{3}(M{=}1,\ N{=}0)$

.	u	d	s
u	a_{11}	a_{12}	a_{13}
d	a_{21}	a_{22}	a_{23}
s	a_{31}	a_{32}	a_{33}

A similar treatment for the IR $\underline{3}^*$, using the corresponding table, given below, gives us the $SU(3)$ matrix $A^\dagger$.

$\underline{3}^*(M{=}0,\ N{=}1)$

.	P	Q	R	S	U	V	I	I_3	Y	$\|PQRSTUV)$	$N^{1/2}$
$\bar{d}$	1	0	0	1	0	0	1/2	1/2	-1/3	w_2	$\sqrt{2}$
$\bar{u}$	0	1	0	0	0	0	1/2	$-1/2$	-1/3	-w_1	$\sqrt{2}$
$\bar{s}$	0	0	0	0	0	1	0	0	2/3	w_3	$\sqrt{2}$

$\underline{3}^*(M{=}0,\ N{=}1)$

.	$\bar{d}$	$\bar{u}$	$\bar{s}$
$\bar{d}$	a_{11}^*	a_{21}^*	a_{31}^*
$\bar{u}$	a_{12}^*	a_{22}^*	a_{32}^*
$\bar{s}$	a_{13}^*	a_{23}^*	a_{33}^*

We now treat the case of the IR $\underline{8}$. The terms relevant for this IR are quadratic in the primed and doubly primed composite variables. For example the first term in the expansion of the generating function Eq.(38) is

$$r'_p r''_p s'_p s''_p = p'^2 r' s' \Big(-A_{11}A_{11}^* pqrs + A_{11}A_{12}^* A_{12}^* p^2 rs + A_{11}A_{13}^* prv - A_{21}A_{12}^* q^2 rs$$

$$+ A_{21}A_{12}^* pqrs + A_{21}A_{13}^* qrv + A_{31}A_{11}^* qsu + A_{31}A_{12}^* psu + A_{31}A_{13}^* uv \Big) \ . \tag{A.2}$$

In Eq.(A.2) the various monomials $p^P q^Q r^R s^S u^U v^V$ correspond to the quantum numbers P, Q, R, S, U, V in the first row of the table corresponding to the IR $\underline{8}$ as indicated in the table given below. Therefore their coefficients give us the first row of the $SU(3)$ Wigner's

D-matrix for the IR $\underline{8}$. One can build the remaining rows in a similar manner. The result is given in the form of a table below.

$\underline{8}(M{=}1,\ N{=}1)$

.	P	Q	R	S	U	V	I	I_3	Y	$\|PQRSTUV)$	$N^{1/2}$
π^+	2	0	1	1	0	0	1	1	0	z_1w_2	$\sqrt{6}$
π^0	1	1	1	1	0	0	1	0	0	$-z_1w_1+z_2w_2$	$\sqrt{12}$
π^-	0	2	1	1	0	0	1	-1	0	$-z_2w_1$	$\sqrt{6}$
K^+	1	0	1	0	0	1	1/2	1/2	1	z_1w_3	$\sqrt{6}$
K^0	0	1	1	0	0	1	1/2	-1/2	1	z_2w_3	$\sqrt{6}$
$\bar{K}^0$	1	0	0	1	1	0	1/2	1/2	-1	w_2z_3	$\sqrt{6}$
K^-	0	1	0	1	1	0	1/2	-1/2	-1	$-w_1z_3$	$\sqrt{6}$
η	0	0	0	0	1	1	0	0	0	$(z_3w_3=-z_1w_1-z_2w_2)$	2

$\underline{8}^*(M{=}1,\ N{=}1)$

.	π^+	π^0	π^-	K^+	K^0	$\bar{K}^0$	K^-	η
π^+	$(a_{21}a_{12}^*-a_{11}a_{11}^*)$	$\frac{a_{11}a_{12}^*}{\sqrt{2}}$	$a_{11}a_{13}^*$	$\frac{-\sqrt{2}a_{21}a_{12}^*}{3}$	$\frac{\sqrt{2}a_{21}a_{13}^*}{3}$	$-\sqrt{2}a_{31}a_{11}^*$	$\sqrt{2}a_{31}a_{12}^*$	$\frac{a_{31}a_{13}^*}{\sqrt{3}}$
π^0	$\frac{(a_{21}a_{21}^*-a_{11}a_{11}^*)}{\sqrt{2}}$	$\frac{a_{11}a_{21}^*}{2}$	$\frac{a_{11}a_{31}^*}{\sqrt{2}}$	$-\frac{a_{21}a_{11}^*}{6\sqrt{2}}$	$\frac{a_{21}a_{31}^*}{6\sqrt{2}}$	$-\frac{a_{31}a_{11}^*}{\sqrt{2}}$	$\frac{a_{31}a_{21}^*}{\sqrt{2}}$	$\frac{a_{31}a_{31}^*}{2\sqrt{3}}$
π^-	$(a_{22}a_{21}^*-a_{12}a_{11}^*)$	$\frac{a_{12}a_{21}^*}{\sqrt{2}}$	$a_{12}a_{31}^*$	$-\frac{\sqrt{2}a_{22}a_{11}^*}{3}$	$\frac{\sqrt{2}a_{22}a_{31}^*}{3}$	$-\sqrt{2}a_{32}a_{11}^*$	$\sqrt{2}a_{32}a_{21}^*$	$\frac{a_{32}a_{31}^*}{\sqrt{3}}$
K^+	$(a_{21}a_{23}^*-a_{11}a_{13}^*)$	$\frac{a_{11}a_{23}^*}{\sqrt{2}}$	$\frac{a_{11}a_{33}^*}{\sqrt{2}}$	$-\frac{a_{21}a_{13}^*}{3}$	$\frac{a_{21}a_{33}^*}{3}$	$-a_{31}a_{13}^*$	$a_{31}a_{23}^*$	$\frac{a_{31}a_{33}^*}{\sqrt{6}}$
K^0	$(a_{21}a_{23}^*-a_{12}a_{13}^*)$	$\frac{a_{12}a_{23}^*}{\sqrt{2}}$	$a_{12}a_{33}^*$	$-\frac{a_{21}a_{13}^*}{3}$	$\frac{a_{21}a_{33}^*}{3}$	$-a_{31}a_{13}^*$	$a_{31}a_{23}^*$	$\frac{a_{31}a_{33}^*}{\sqrt{6}}$
$\bar{K}^0$	$(a_{23}a_{22}^*-a_{13}a_{12}^*)$	$\frac{a_{13}a_{22}^*}{\sqrt{2}}$	$a_{13}a_{32}^*$	$-\frac{a_{23}a_{12}^*}{3}$	$\frac{a_{23}a_{32}^*}{3}$	$-a_{33}a_{12}^*$	$a_{33}a_{22}^*$	$\frac{a_{33}a_{32}^*}{\sqrt{6}}$
K^-	$(a_{23}a_{21}^*-a_{13}a_{11}^*)$	$\frac{a_{13}a_{21}^*}{\sqrt{2}}$	$a_{13}a_{31}^*$	$-\frac{a_{23}a_{11}^*}{3}$	$\frac{a_{23}a_{31}^*}{3}$	$-a_{33}a_{11}^*$	$a_{33}a_{21}^*$	$\frac{a_{33}a_{31}^*}{\sqrt{6}}$
η	$\frac{\sqrt{3}(a_{23}a_{23}^*-a_{13}a_{13}^*)}{\sqrt{2}}$	$\frac{\sqrt{3}a_{13}a_{23}^*}{2}$	$\frac{\sqrt{3}a_{13}a_{33}^*}{\sqrt{2}}$	$-\frac{a_{23}a_{13}^*}{\sqrt{6}}$	$\frac{a_{23}a_{33}^*}{\sqrt{6}}$	$-\frac{\sqrt{3}a_{33}a_{13}^*}{\sqrt{2}}$	$\frac{\sqrt{3}a_{33}a_{23}^*}{\sqrt{2}}$	$\frac{a_{33}a_{33}^*}{2}$

In all the above computations a normalization factor for each D-matrix element is computed with the help of the Eq.(28).

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